McGill University
Department of Electrical and Computer Engineering

Communication Systems (304-411)

These notes do not replace class notes.

These notes are designed as a teaching aid for self review of the material, and they should be used together with the class notes as well as the text book. The students are encouraged to read these notes and follow the instructions.

1 Introduction

1.1 The communication process

Transport of information between points separated in space and/or time.

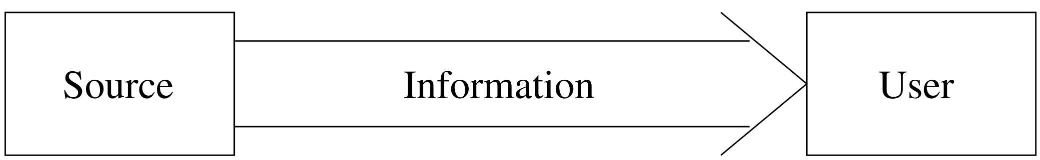


Figure 1: A communication process

Examples :

- Points separated in space :
  - telephone (point to point)
  - broadcast radio and TV (point to multi-point)
  - cellular systems (point to multi-point in down link, multi-point to point in the up link).
- Points separated in time :
  - magnetic recording systems
  - compact disk systems.

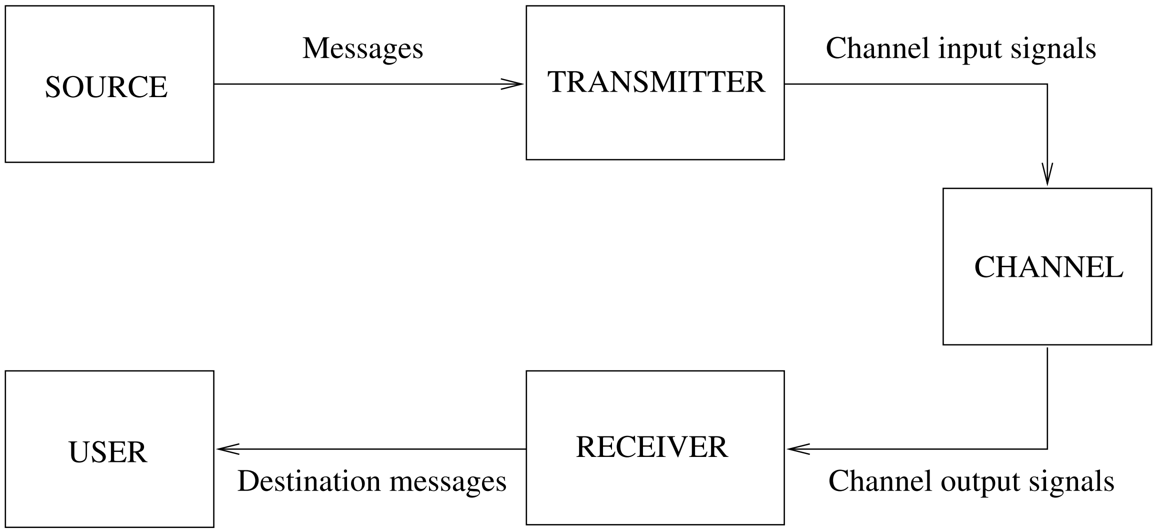


Figure 2: General model for a communication system

Transmitter : Source messages → Channel input signals
Channel : Channel input signals → Channel output signals
Receiver : Channel output signals → User messages

Communication systems design :
Given : Source, Channel and User, find a Transmitter and Receiver such that the user messages are ”as close as possible” to source messages.
Difference between source message and user message is called distortion.
Objective : small distortion
Resources : transmission power, bandwidth, complexity, and possible others depending on specific systems.

1.2 Detailed model of a communication system

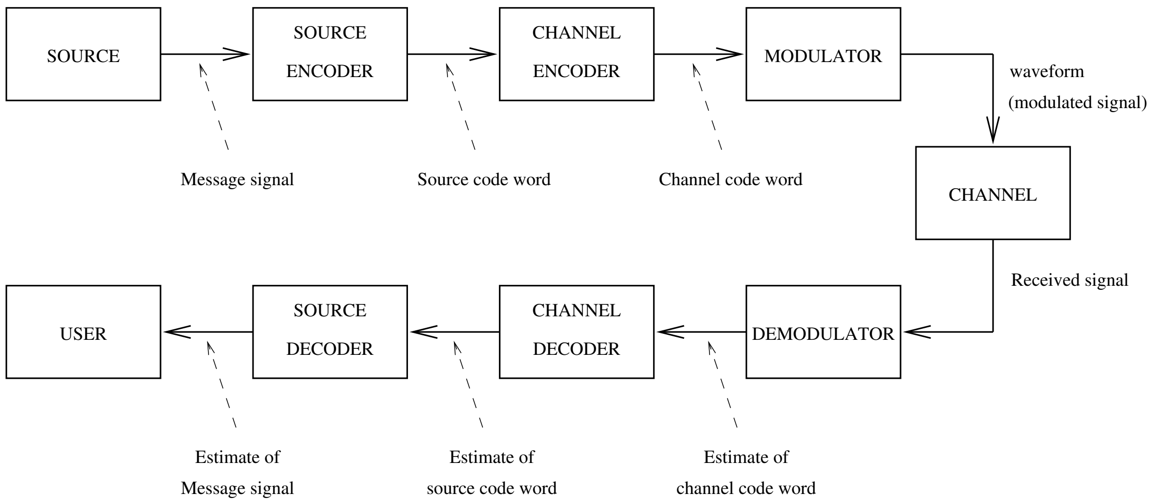


Figure 3: General model for a communication system

Source encoder :
Removes unnecessary details (redundancies, ets.) from source messages, and keeps only the information that needs to be transmitted.

Channel encoder :
Adds redundancy to the information that has to be transmitted, in order to reduce the effects of channel noise.

Modulator :
Maps the transmitted information into channel input signals.

Demodulator + Channel decoder :
Attempt to recover the transmitted information from the channel output (distorted) signal.

Source decoder :
Converts the recovered information to a form suitable for user.

Source or Channel encoder may or may not be needed. Modulators are always needed.

1.3 Analog and digital systems

Analog modulation: The set of all possible channel input signals is continuous (infinite).

*Digital modulation:*The set of all possible channel input signals is discrete and finite.

Examples :

Analog Communication Systems :
Suppose that the value of a variable A has to be transmitted. It is known that $0 < A < 1$.
The modulation procedure could be $x(t) = A\cos(\omega_c t)$ where $\omega_c = 2\pi f_c$. In this case the waveform $x(t)$ is transmitted over the channel. Other modulation scheme could employ $x(t) = \cos(\omega_c t + 2\pi A)$, or in general $x(t) = s(t, A)$ where $s(t, A)$ is a time function that depends on the parameter A.

Digital Communication Systems:
Suppose that it is known that $A = 1/4$ or $A = 1/2$. The modulation procedure could be :

$$x(t) = \begin{cases} 1/4\cos(\omega_c t) & , \quad \text{if } A = 1/4 \\ 1/2\cos(\omega_c t) & , \quad \text{if } A = 1/2 \end{cases}$$

or :

$$x(t) = \begin{cases} \cos[(\omega_c - \Delta)t] & , \quad \text{if } A = 1/4 \\ \cos[(\omega_c + \Delta)t] & , \quad \text{if } A = 1/2 \end{cases}$$

or :

$$x(t) = \begin{cases} s_1(t) & , \quad \text{if } A = 1/4 \\ s_2(t) & , \quad \text{if } A = 1/2 \end{cases}$$

where $s_1(t) \neq s_2(t)$.

2 Representation of signals and systems

2.1 Time and frequency domains (review)

a) Fourier transform

Give the definition of the Fourier transform, and state the conditions for its existance.

b) The Dirac delta function

Definitions:

- The delta function $\delta(t)$ is defined as the generalized function or distribution that satisfies

  For any continuous function $f(t)$ at $t = 0$ $$\int_{t_1}^{t_2} f(t)\delta(t)dt = \begin{cases} f(0) & \text{if } t_1 < 0 < t_2 \\ 0 & \text{else.} \end{cases}$$

  Therefore $\delta(t)$ satisfies

- $\int_{-\infty}^{\infty} \delta(t)dt = 1$, suggesting that $\delta(t) = 0\ \forall t \neq 0$. Note that this latter definition is to be used only in the context of integration.

- The delta function can also be considered as the limit of sequences of functions

$$\delta(t) = \lim_{a\to\infty} a\ \text{rect}(at) \qquad \text{where} \qquad \text{rect}(v) = \begin{cases} 1, & \text{for } |v| \leq \frac{1}{2}, \\ 0, & \text{for } |v| > \frac{1}{2} \end{cases}$$

$$= \lim_{a\to\infty} a\ \text{sinc}(at) \qquad \text{where} \qquad \sin(v) = \frac{\sin(\pi v)}{\pi v}$$

$$= \lim_{a\to\infty} a e^{-\pi a^2 t^2}$$

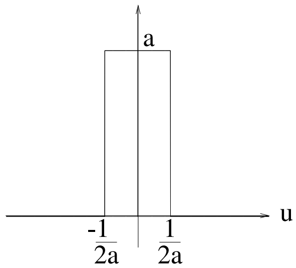
Properties:

- $\delta(t)$ is an even function
- $\delta(t)$ satisfies the sifting or sampling property:

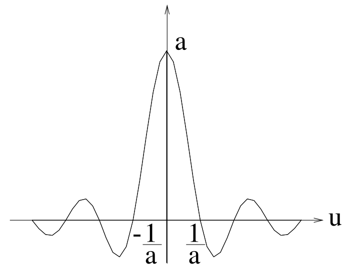
  For any continuous function $f(t)$ at t_0 $$\int_{-\infty}^{\infty} f(t)\delta(t - t_0)dt = f(t_0)$$

- $\delta(t)$ satisfies the replication property:

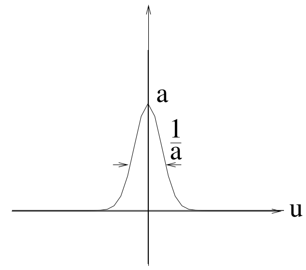
  For any continuous function $f(t)$ at t_0 $$f(t) * \delta(t - t_0) = f(t - t_0)$$



(a) graph of $a\ \mathrm{rect}(at)$



(b) graph of $a\ \mathrm{sinc}(at)$



(c) graph of $ae^{-\pi a^2 t^2}$

- The delta function satisfies

$$\begin{aligned} f(t)\delta(t-t_0) &= f(t_0)\delta(t-t_0) \\ \delta(at) &= \frac{1}{|a|}\delta(t) \qquad a \neq 0 \\ \delta(t) &= \frac{du(t)}{dt} \end{aligned}$$

where $u(t)$ is the unit step function. (**To be rigorous these equations are valid only if they appear inside integrals.**)

- Fourier transform relationships:

$$\delta(t) \xrightarrow{\mathcal{F}} 1 \qquad \text{since} \quad \mathcal{F}\left[\delta(t)\right] = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft} dt = \left[e^{-j2\pi ft}\right]_{t=0} = 1$$

$$1 \xrightarrow{\mathcal{F}} \delta(-f) = \delta(f) \qquad \text{by the frequency duality property of the Fourier transform}$$

Therefore $\int_{-\infty}^{\infty} e^{-j2\pi ft} dt = \delta(f)$ and

$$\begin{aligned} e^{j2\pi f_c t} &\xrightarrow{\mathcal{F}} \delta(f-f_c) \\ \cos 2\pi f_c t &\xrightarrow{\mathcal{F}} \frac{1}{2}\big[\delta(f-f_c)+\delta(f+f_c)\big] \\ \sin 2\pi f_c t &\xrightarrow{\mathcal{F}} \frac{1}{2j}\big[\delta(f-f_c)-\delta(f+f_c)\big] \end{aligned}$$

2.2 Hilbert transform and complex envelope representation of signals

a) Hilbert transform

Let $x(t)$ be a real signal, the Hilbert transform of $x(t)$ is defined by

$$\hat{x}(t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau}d\tau = \lim_{\substack{\epsilon\to 0\\ A\to\infty}} \frac{1}{\pi}\left[\int_{-A}^{t-\epsilon} \frac{x(\tau)}{t-\tau}d\tau + \int_{t+\epsilon}^{A} \frac{x(\tau)}{t-\tau}d\tau\right]$$

Cauchy's principal value of the integral

$$\hat{x}(t) = x(t) * \frac{1}{\pi t}$$

Fourier transform of $\hat{x}(t)$:

$$\mathcal{F}\left\{\hat{x}(t)\right\} = \hat{X}(f) = \mathcal{F}\left\{x(t)\right\} \cdot \mathcal{F}\left\{\frac{1}{\pi t}\right\} = X(f)\left(-j\text{sgn}(f)\right) = -j\text{sgn}(f)X(f)$$

where $\text{sgn}(f)$ is given by

$$\text{sgn}(f) = \begin{cases} 1, & f > 0 \\ 0, & f = 0 \\ -1, & f < 0 \end{cases}$$

Let $H(f) = -j\text{sgn}(f)$ (Hilbert transformer)

Positive frequencies: phase shift of -90^o (90^o phase lag).

Negative frequencies: phase shift of 90^o (90^o phase lead).

Inverse Hilbert transform :

Let us assume that $X(0) = 0$,

$$\hat{\hat{X}}(f) = -j\,\text{sgn}(f)\hat{X}(f) = \left[-j\,\text{sgn}(f)\right]^2 X(f) = -X(f)$$

Thus

$$\begin{aligned} \widehat{\hat{x}(t)} &= -x(t) \\ x(t) = -\widehat{\hat{x}(t)} &= -\hat{x}(t) * \frac{1}{\pi t} = -\frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\hat{x}(\tau)}{t-\tau}d\tau \end{aligned}$$

Hilbert transform of cosine and sine functions

Let $x(t) = \cos 2\pi f_0 t$, then $X(f) = \frac{1}{2}\big[\delta(f - f_0) + \delta(f + f_0)\big]$.

$$\begin{aligned}\hat{X}(f) &= -j\,\text{sgn}(f)X(f) = -\frac{j}{2}\big[\text{sgn}(f_0)\delta(f - f_0) + \text{sgn}(-f_0)\delta(f + f_0)\big] \\ &= -\frac{j}{2}\big[\delta(f - f_0) - \delta(f + f_0)\big]\end{aligned}$$

Thus

$$\hat{x}(t) = \sin 2\pi f_0 t$$

Similarly show that if $x(t) = \sin 2\pi f_0 t$, then $\hat{x}(t) = -\cos 2\pi f_0 t$.

Theorem 2.1 (Hilbert transform of the product of a low pass and a high pass signal). *Suppose that $x(t) = f(t)g(t)$ where $f(t)$ is a low pass (i.e. $F(f) = 0, |f| > W$) and $g(t)$ is a high pass signal (i.e. $G(f) = 0, |f| < W$), and $f(f)$ and $G(f)$ do not overlap, then $\hat{x}(t) = f(t)\hat{g}(t)$.*

Exercise: Draw an example of the spectrum of a low pass and high pass signals.

Proof:

$$\begin{aligned}X(f) &= \int_{-\infty}^{\infty} F(f - u)G(u)du \\ \hat{X}(f) &= -j\ \text{sgn}(f)\int_{-\infty}^{\infty} F(f - u)G(u)du\end{aligned}$$

$$\begin{aligned}\mathcal{F}\big[f(t)\hat{g}(t)\big] = F(f) * \hat{G}(f) &= \int_{-\infty}^{\infty} F(f - u) \cdot -j\,\text{sgn}(u)G(u)du \\ \hat{X}(f) - \mathcal{F}\left[f(t)\hat{g}(t)\right] &= -j\int_{-\infty}^{\infty} F(f - u)G(u)\big[\text{sgn}(f) - \ \text{sgn}(u)\big]du = 0\end{aligned}$$

since $F(u)$ and $G(u)$ do not overlap.

b) Pre-envelope or complex analytic (CA) signal

The pre-envelope or complex analytic (CA) signal associated with a real-valued signal $x(t)$ is defined by

$$x_+(t) = x(t) + j\hat{x}(t)$$

where $\hat{x}(t)$ is the Hilbert transform of $x(t)$. The pre-envelope satisfies

$$x(t) = \Re\{x_+(t)\}$$

The Fourier transform of $x_+(t)$ is given by

$$\begin{aligned}\mathcal{F}\{x_+(t)\} = X_+(f) &= X(f) + j\hat{X}(f) \\ &= X(f) + j(-j \text{ sgn}(f))X(f) \\ &= X(f)\,[1 + \text{ sgn}(f)] \\ &= \begin{cases} \qquad & f > 0 \\ \qquad & f = 0 \\ \qquad & f < 0 \end{cases} \qquad \textit{exercise: complete the blanks}\end{aligned}$$

Exercise: Draw the Fourier transform of the pre-envelope of a low pass signal.

b) Complex envelope (CE) of $x(t)$

Let $x(t)$ be a real signal then the complex envelope (CE) of $x(t)$ with respect to the carrier f_0 is defined as

$$\tilde{x}(t) = x_+(t)e^{-j2\pi f_0 t} = \big(x(t) + j\hat{x}(t)\big)e^{-j2\pi f_0 t}$$

and we have

$$x(t) = \Re\left\{\tilde{x}(t)e^{j2\pi f_0 t}\right\}$$

Example: Find the complex envelope with respect to f_0 of an RF pulse $x(t) = A\text{rect}\left(\frac{t}{T}\right)\cos 2\pi f_c t$, where $f_c \neq f_0$.

Frequency domain relations

$$x(t) = \Re\left\{\tilde{x}(t)e^{j2\pi f_0 t}\right\} = \frac{1}{2}\tilde{x}(t)e^{j2\pi f_0 t} + \frac{1}{2}\tilde{x}^*(t)e^{-j2\pi f_0 t}$$

Let $\tilde{X}(f) = \mathcal{F}\{\tilde{x}(t)\}$, then

$$X(f) = \frac{1}{2}\tilde{X}(f) * \delta(f - f_0) + \frac{1}{2}\tilde{X}^*(-f) * \delta(f + f_0)$$
$$= \frac{1}{2}\tilde{X}(f - f_0) + \frac{1}{2}\tilde{X}^*(-f - f_0)$$

Since $\tilde{x}(t) = x_+(t)e^{-j2\pi f_0 t}$,

$$\tilde{X}(f) = X_+(f) * \delta(f + f_0) = X_+(f + f_0) = \begin{cases} 2X(f + f_0) & , \quad \text{if } f > -f_0 \\ X(0) & , \quad \text{if } f = -f_0 \\ 0 & , \quad \text{if } f < -f_0 \end{cases}$$

Exercise: Assume that $x(t)$ is a low pass signal, draw the Fourier transform of its complex envelope with respect to f_0. Repeat assuming that $x(t)$ is a bandpass signal centered around $\pm f_c$. Assume $f_c \neq f_0$.

d) bandpass signals

Let $x(t)$ be a real signal such that $X(f)$ is non-zero (or non-negligible) only in some frequency band centered about $\pm f_c$ and $f_c \gg 0$, then $x(t)$ is a **bandpass signal**.

Exercise: Draw an example of the spectrum of a bandpass signal of bandwidth $2W$.

If $2W \ll f_c$ then $x(t)$ is called a **narrow-band** signal.

Exercise: Show that the complex envelope with respect to f_c of a bandpass signal centered about $\pm f_c$ is a low-pass signal.

Since $\tilde{x}(t)$ is a complex-valued function

$$\tilde{x}(t) = x_I(t) + jx_Q(t)$$

where $x_I(t)$ and $x_Q(t)$ are real-valued low-pass functions.

Representation of a bandpass signal:

By definition of the complex envelope

$$\begin{aligned} x(t) &= \Re\left\{\tilde{x}(t)e^{j2\pi f_c t}\right\} \\ &= \Re\left\{\left[x_I(t) + jx_Q(t)\right]e^{j2\pi f_c t}\right\} \\ &= x_I(t)\cos 2\pi f_c t - x_Q(t)\sin 2\pi f_c t \end{aligned}$$

where $x_I(t)$ and $x_Q(t)$ are low-pass and real, i.e.

$$\begin{aligned} X_I(f) &= 0, \quad |f| > W \quad \text{and } W < f_c \\ X_Q(f) &= 0, \quad |f| > W \end{aligned}$$

$x_I(t)$ is referred as the **in-phase** component of $x(t)$, $x_Q(t)$ is referred as the **quadrature** component of $x(t)$.

$$
\begin{aligned}
x_+(t) &= x(t) + j\hat{x}(t) \\
&= x_I(t)\cos 2\pi f_c t - x_Q(t)\sin 2\pi f_c t + j\ \widehat{x_I(t)\cos 2\pi f_c t}\ - j\ \widehat{x_Q(t)\sin 2\pi f_c t} \\
&= x_I(t)\cos 2\pi f_c t - x_Q(t)\sin 2\pi f_c t + jx_I(t)\ \widehat{\cos 2\pi f_c t}\ - jx_Q(t)\ \widehat{\sin 2\pi f_c t} \\
&= x_I(t)\cos 2\pi f_c t - x_Q(t)\sin 2\pi f_c t + jx_I(t)\sin 2\pi f_c t + jx_Q(t)\cos 2\pi f_c t \\
&= [x_I(t) + jx_Q(t)]\, e^{j2\pi f_c t} \\
&= \tilde{x}(t)e^{j2\pi f_c t}
\end{aligned}
$$

which is consistent with the definition of $\tilde{x}(t)$.

complex envelope of $x(t)$ (with respect to f_c):

$$\tilde{x}(t) = x_I(t) + jx_Q(t)$$

(natural) envelope of $x(t)$:

$$r(t) = |\tilde{x}(t)| = \sqrt{x_I^2(t) + x_Q^2(t)}$$

phase of $x(t)$:

$$\phi(t) = \arg\left[\tilde{x}(t)\right] = \tan^{-1}\left(\frac{x_Q(t)}{x_I(t)}\right)$$

Therefore

$$
\begin{aligned}
\tilde{x}(t) &= r(t)e^{j\phi(t)} \\
x(t) &= \Re\left\{\tilde{x}(t)e^{j2\pi f_c t}\right\} \\
&= \Re\left\{r(t)e^{j(2\pi f_c t + \phi(t))}\right\} \\
&= r(t)\cos\bigl(2\pi f_c t + \phi(t)\bigr)
\end{aligned}
$$

Exercise: Give an interpretation of the complex envelope of $x(t)$ as a time-varying phasor.

Fourier transform of bandpass signals

$$X_I(f) = \mathcal{F}\{x_I(t)\} \qquad X_Q(f) = \mathcal{F}\{x_Q(t)\} \qquad X(f) = \mathcal{F}\{x(t)\}$$

$$\begin{aligned}
x(t) &= x_I(t)\cos 2\pi f_c t - x_Q(t)\sin 2\pi f_c t \\
X(f) &= X_I(f) * \frac{1}{2}\left[\delta(f-f_c) + \delta(f+f_c)\right] - X_Q(f) * \frac{1}{2j}\left[\delta(f-f_c) - \delta(f+f_c)\right] \\
&= \frac{1}{2}X_I(f-f_c) + \frac{1}{2}X_I(f+f_c) + \frac{j}{2}X_Q(f-f_c) - \frac{j}{2}X_Q(f+f_c) \\
&= \frac{1}{2}\left[X_I(f-f_c) + jX_Q(f-f_c)\right] + \frac{1}{2}\left[X_I(f+f_c) - jX_Q(f+f_c)\right]
\end{aligned}$$

since $x_I(t)$ and $x_Q(t)$ are real. Hence

$$X(f) = \frac{1}{2}\tilde{X}(f-f_c) + \frac{1}{2}\tilde{X}^*(-f-f_c)$$

Note that although $\tilde{X}(f) = X_I(f) + jX_Q(f)$, we have $X_I(f) \neq \Re\left\{\tilde{X}(f)\right\}$ and $X_Q(f) \neq \Im\left\{\tilde{X}(f)\right\}$, since $X_I(f)$ and $X_Q(f)$ are not necessarily real.

Exercise: Show that

$$\begin{aligned}
X_I(f) &= \frac{1}{2}\left[X_+(f+f_c) + X_+^*(-f+f_c)\right] \\
X_Q(f) &= \frac{1}{2j}\left[X_+(f+f_c) - X_+^*(-f+f_c)\right]
\end{aligned}$$

Generation of a bandpass signal from $x_I(t)$ and $x_Q(t)$: Transformation up frequency conversion (UFC)

Draw a block diagram that generates a bandpass signal from $x_I(t)$ and $x_Q(t)$.

Generation of the complex envelope of a signal $x(t)$: Transformation down frequency conversion (DFC)

Draw a block diagram that generates the complex envelope of a signal $x(t)$.

e) Bandpass linear time-invariant systems

A linear time-invariant system is called bandpass if its impulse response is a bandpass signal.

Let $x(t)$ be a bandpass signal applied to the input of a linear time-invariant bandpass system (around the same carrier frequency), then the output $y(t) = h(t) * x(t)$ is also a bandpass signal and

$$\tilde{y}(t) = \frac{1}{2}\tilde{h}(t) * \tilde{x}(t)$$

where $\tilde{x}(t)$, $\tilde{h}(t)$ and $\tilde{y}(t)$ are the complex envelopes of $x(t)$, $h(t)$ and $y(t)$ with respect to the same carrier.

Exercise : Prove this result by first showing that $y(t)$ is a bandpass signal. Then prove the relationship by manipulating $y(t)$ using pre-envelopes so that it is in the form

$$y(t) = \Re\left\{XXe^{j2\pi f_c t}\right\}$$

where XX is to be found. By definition XX is the complex envelope of $y(t)$, so the result follows.