5 Analog carrier modulation with noise

5.1 Noisy receiver model

Assume that the modulated signal $x(t)$ is passed through an additive White Gaussian noise channel. A noisy receiver model is illustrated in Fig. 16.

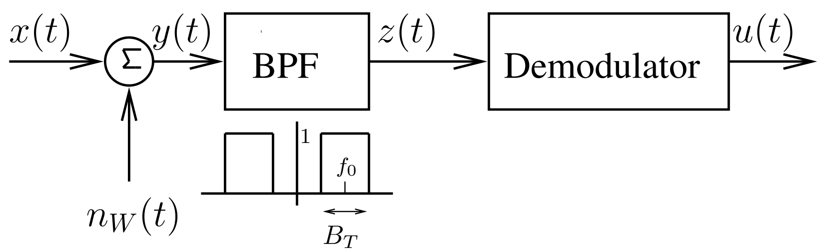

Figure 16: Noisy receiver model

$n_W(t)$ is the realization of a white random process $N_W(t)$ with power spectral density $\frac{N_0}{2}$ Watts/Hz, i.e.

$$E\left[N_W(t)N_W(u)\right] = \frac{N_0}{2}\delta(t-u) \qquad S_{N_W}(f) = \frac{N_0}{2}$$

The bandpass filter is designed to pass the modulated signal undistorted (and also amplify it in practice) but will cut noise power outside the frequency band of the modulated signal.

The input of the bandpass filter is given by

$$y(t) = x(t) + n_W(t)$$

The output of the bandpass filter is given by

$$z(t) = x(t) + n(t)$$

where $n(t)$ is the realization of the bandpass random process $N(t)$.
Exercise: Draw the power spectral density of $N_W(t)$ and show that $N_W(t)$ has an infinite power.

Exercise: Draw the power spectral density of $N(t)$ and show that $N(t)$ is a band-limited random process (band-limited noise) of power $E\left[N^2(t)\right] = R_N(0) = B_T N_0$.

Hence it is seen that the bandpass filter reduces the noise power.

We assume that the message signal $m(t)$ is the realization of a stationary random process of zero mean, whose power spectral density $S_M(f)$ is limited to a maximum frequency W. W is referred as the **message bandwidth**. Let $M(t)$ be the corresponding random process.

Example of the spectrum of $M(t)$: (draw an example)

For the modulation studied in this course (AM, DSB-SC, SSB, VSB, FM, PM), the modulated signal $X(t)$ is a bandpass random process with bandwidth B_T, whose spectrum is centered around f_0 with realization denoted $x(t)$. f_0 is defined as follows

$$f_0 = \begin{cases} f_c, & \text{AM,DSB-SC, FM, PM} \\ f_c - \frac{W}{2}, & \text{lower side-band SSB} \\ f_c + \frac{W}{2}, & \text{upper side-band SSB} \end{cases}$$

Example of the spectrum of $X(t)$: (draw an example)

Assessing noise performance:

Assume that the output of the demodulator can be written as

$$u(t) = \breve{m}(t) + w(t)$$

where $\breve{m}(t)$ is a realization of the recovered message signal and $w(t)$ is the noise component at the output of the demodulator. The effect of $w(t)$ can be assessed by calculating the **Output-Signal-to-noise ratio**.

$$\text{SNR}_0 = \frac{\text{Average power of the demodulated message signal}}{\text{Average power of the noise measured at the receiver output}} = \frac{P_{\breve{M}}}{P_W}$$

The output SNR is meaningful only if there is an additive relationship between the signal and noise at the receiver output. So for FM or AM with envelope detection, we will consider low noise power at the receiver input to obtain an approximate additive relationship.

Since the output SNR depends on the modulation and the demodulation techniques used, to obtain a fair comparison of several demodulation techniques, we should use

$$\frac{\mathrm{SNR}_0}{\mathrm{SNR}_I}$$

where SNR_I is the **Input-Signal-to-noise ratio** defined as

$$\mathrm{SNR}_I = \frac{\text{Average power of the modulated message signal (at demodulator input)}}{\text{Average power of the noise measured at the demodulator input}} = \frac{P_R}{P_N} = \frac{P_X}{N_0 B_T}$$

P_R : Signal power at the receiver input (in absence of noise) = noiseless signal power at demodulator input
P_N: Average power of the filtered noise $N(t)$

Since P_N depends on the modulation through its transmission bandwidth, to obtain a fair comparison of noise performance for several modulation schemes, SNR_I is not appropriate, and the **baseband (channel) signal-to-noise ratio** $\mathrm{SNR}_b = \mathrm{SNR}_C$ should be used, where

$$\mathrm{SNR}_b = \mathrm{SNR}_C = \text{Signal-to-noise ratio without modulation}$$

5.2 Baseband reference system

A reference baseband system is obtained by passing a noisy message signal of power equal to the power of the modulated signal through a low-pass filter of bandwidth equal to that of the message signal as illustrated in Fig. 17.

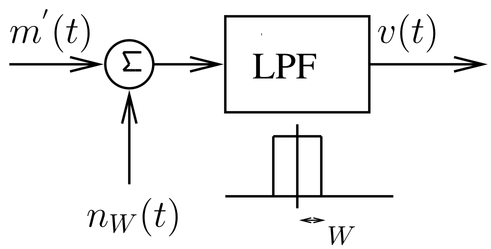

Figure 17: Baseband reference system

The output of the low-pass filter is given by

$$v(t) = m'(t) + n'(t)$$

where $m'(t)$ has the same power as the modulated wave. Then

$$\mathrm{SNR}_b = \mathrm{SNR}_C = \frac{\text{Average power of the modulated signal}}{\text{Average power of the noise in the message bandwidth}} = \frac{P_{M'}}{P_{N'}} = \frac{P_R}{N_0 W}$$

5.3 SNR's for linear modulations with coherent detection

Exercise: Draw a block diagram of a noisy receiver model with coherent detection.

$$
\begin{aligned}
y(t) &= x(t) + n_W(t) \quad \text{(Received signal)} \\
z(t) &= x(t) + n(t) \quad \text{(Output of bandpass filter)} \\
x(t) &= x_I(t)\cos(2\pi f_c t) - x_Q(t)\sin(2\pi f_c t) \\
n(t) &= n_I(t)\cos(2\pi f_0 t) - n_Q(t)\sin(2\pi f_0 t) \quad \text{Rice representation}
\end{aligned}
\tag{16}
$$

Note that lowercase letters denote realizations of the random processes denoted by capital letters.

Exercise: Draw the power spectral densities of $N_I(t)$ and $N_Q(t)$

a) Input signal power: P_R

Let us assume that $X_I(t)$ and $X_Q(t)$ are jointly wide-sense stationary with constant means K_I and K_Q.

Exercise: Verify that for AM, DSB-SC, SSB and VSB $X_I(t)$ and $X_Q(t)$ are jointly wide-sense stationary with constant mean.

Hint: Recall that $M(t)$ is assumed to a WSS random process with zero mean and WSS stationarity is preserved by linear time-invariant filtering. Note that $X_I(t)$ is zero mean except for AM modulation due to the DC component.

To find the received power P_R, we set $n_W(t) = 0$ in (16) and calculate the power of the signal left. If $n_W(t) = 0$ then

$$n(t) = 0 \quad \Longrightarrow z(t) = x(t)$$

Hence

$$P_R = P_Z = P_X = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} E\left[X^2(t)\right] dt$$

In order to simplify P_R let us show that $X(t)$ is a wide-sense cyclo stationary random process.

$$E\left[X(t)\right] = E\left[X_I(t)\right]\cos(2\pi f_c t) - E\left[X_Q(t)\right]\sin(2\pi f_c t) = K_I\cos(2\pi f_c t) - K_Q\sin(2\pi f_c t)$$

where K_I and K_Q are constants. Therefore the mean of $X(t)$ is periodic with period $1/f_c$.

$$\begin{aligned}
E\left[X(t+\tau)X(t)\right] &= E\left[\big(X_I(t+\tau)\cos(2\pi f_c(t+\tau)) - X_Q(t+\tau)\sin(2\pi f_c(t+\tau))\big)\cdot\right.\\
&\qquad \left.\big(X_I(t)\cos(2\pi f_c t) - X_Q(t)\sin(2\pi f_c t)\big)\right]\\
&= R_{X_I}(\tau)\cos(2\pi f_c(t+\tau))\cos(2\pi f_c t) - R_{X_I X_Q}(\tau)\cos(2\pi f_c(t+\tau))\sin(2\pi f_c t)\\
&\quad - R_{X_Q X_I}(\tau)\sin(2\pi f_c(t+\tau))\cos(2\pi f_c t) + R_{X_Q}(\tau)\sin(2\pi f_c(t+\tau))\sin(2\pi f_c t)\\
&= \frac{1}{2}R_{X_I}(\tau)\left[\cos(2\pi f_c(2t+\tau)) + \cos(2\pi f_c\tau)\right]\\
&\quad - \frac{1}{2}R_{X_I X_Q}(\tau)\left[\sin(2\pi f_c(2t+\tau)) - \sin(2\pi f_c\tau)\right]\\
&\quad - \frac{1}{2}R_{X_Q X_I}(\tau)\left[\sin(2\pi f_c(2t+\tau)) + \sin(2\pi f_c\tau)\right]\\
&\quad - \frac{1}{2}R_{X_Q}(\tau)\left[\cos(2\pi f_c(2t+\tau)) - \cos(2\pi f_c\tau)\right]
\end{aligned}$$

Hence it is seen that $E\left[X(t+\tau)X(t)\right]$ is periodic with period $1/2f_c$. Hence the common period is $1/f_c$ and $X(t)$ is wide-sense cyclo stationary with power defined as

$$\begin{aligned}
P_X = R_X^a(0) &= \frac{1}{2f_c}\int_{-f_c}^{f_c} E\left[X^2(t)\right] dt\\
&= \frac{1}{2}R_{X_I}(0)\cos(2\pi f_c 0) + \frac{1}{2}R_{X_I X_Q}(0)\sin(2\pi f_c 0) - \frac{1}{2}R_{X_Q X_I}(0)\sin(2\pi f_c 0) + \frac{1}{2}R_{X_Q}(0)\cos(2\pi f_c 0)\\
&= \frac{1}{2}R_{X_I}(0) + \frac{1}{2}R_{X_Q}(0) \qquad (17)
\end{aligned}$$

Note that (17) is only valid if $X_I(t)$ and $X_Q(t)$ are jointly wide-sense stationary with constant means.

b) Input noise power: P_N

$$P_N = N_0 B_T$$

c) Input signal-to-noise ratio: SNR_I

$$\mathrm{SNR}_I = \frac{P_R}{N_0 B_T} = \frac{W}{B_T}\mathrm{SNR}_C$$

Recall that random processes with signal components are denoted by capital letters and their realizations are denoted by lowercase letters.

AM:

$$x_I(t) = A_c\left(1 + \mu m_n(t)\right) \qquad x_Q(t) = 0 \qquad B_T = 2W$$

$$\begin{aligned} R_{X_I}(0) &= E\left[X_I^2(t)\right] = A_c^2 + A_c^2\mu^2 E\left[M_n^2(t)\right] + 2A_c\mu E\left[M_n(t)\right] = A_c^2\left(1 + \mu^2 P_{M_n}\right) \\ R_{X_Q}(0) &= E\left[X_Q^2(t)\right] = 0 \end{aligned}$$

$$\mathrm{SNR}_I = \frac{\frac{1}{2}A_c^2\left(1 + k_a^2 P_M\right)}{2N_0 W} = \frac{A_c^2\left(1 + \mu^2 P_{M_n}\right)}{4N_0 W} = \frac{A_c^2\left(1 + k_a^2 P_M\right)}{4N_0 W}$$

where P_M is the power of $M(t)$, P_{M_n} is the power of $M_n(t) = \frac{M(t)}{\max|m(t)|}$ given by

$$P_{M_n} = E\left[M_n^2(t)\right] = \frac{P_M}{\left(\max|M(t)|\right)^2} = \frac{1}{\left(\max|m(t)|\right)^2}\int_{-\infty}^{\infty} S_M(f)df$$

Note that we assume that $\max|m(t)|$ is the same for all the realizations of $M(t)$. If this condition is not satisfied, then the expression of SNR_I should be expressed in terms of the amplitude sensitivity k_a and P_M instead of the modulation index μ and P_{M_n}.

DSB-SC:

$$x_I(t) = A_c m(t) \qquad x_Q(t) = 0 \qquad B_T = 2W$$

$$\begin{aligned} R_{X_I}(0) &= E\left[X_I^2(t)\right] = A_c^2 E\left[M^2(t)\right] = A_c^2 P_M \\ R_{X_Q}(0) &= E\left[X_Q^2(t)\right] = 0 \end{aligned}$$

$$\mathrm{SNR}_I = \frac{A_c^2 P_M}{4N_0 W}$$

where P_M is the power of $M(t)$ given by

$$P_M = E\left[M^2(t)\right] = \int_{-\infty}^{\infty} S_M(f)df$$

SSB:

$$x_I(t) = \frac{A_c}{2}m(t) \qquad x_Q(t) = \pm\frac{A_c}{2}\hat{m}(t) \qquad B_T = W$$

$$R_{X_I}(0) = E\left[X_I^2(t)\right] = \frac{A_c^2}{4}E\left[M^2(t)\right] = \frac{A_c^2}{4}P_M$$

$$R_{X_Q}(0) = E\left[X_Q^2(t)\right] = \frac{A_c^2}{4}E\left[\hat{M}^2(t)\right] = \frac{A_c^2}{4}R_{\hat{M}}(0) = \frac{A_c^2}{4}R_M(0) = \frac{A_c^2}{4}P_M$$

$$\mathrm{SNR}_I = \frac{\frac{A_c^2}{4}P_M}{N_0 W} = \frac{A_c^2 P_M}{4N_0 W}$$

d) Output signal power

By definition the center frequency of the bandpass filter can be written as

$$f_0 = f_c + \alpha$$

where $-\frac{B_T}{2} \leq \alpha \leq \frac{B_T}{2}$, depending on the modulation format.

The output of the bandpass filter is given by

$$\begin{aligned} z(t) &= x_I(t)\cos(2\pi f_c t) - x_Q(t)\sin(2\pi f_c t) + n_I(t)\cos\big(2\pi(f_c+\alpha)t\big) - n_Q(t)\sin\big(2\pi(f_c+\alpha)t\big) \\ &= \big[x_I(t) + n_I(t)\cos(2\pi\alpha t) - n_Q(t)\sin(2\pi\alpha t)\big]\cos(2\pi f_c t) \\ &\quad - \big[x_Q(t) + n_I(t)\sin(2\pi\alpha t) + n_Q(t)\cos(2\pi\alpha t)\big]\sin(2\pi f_c t) \end{aligned}$$

The output of the multiplier is given by

$$\begin{aligned} v(t) &= z(t)\cos(2\pi f_c t) \\ &= \frac{1}{2}\big[x_I(t) + n_I(t)\cos(2\pi\alpha t) - n_Q(t)\sin(2\pi\alpha t)\big] \\ &\quad + \frac{1}{2}\big[x_I(t) + n_I(t)\cos(2\pi\alpha t) - n_Q(t)\sin(2\pi\alpha t)\big]\cos(4\pi f_c t) \\ &\quad - \frac{1}{2}\big[x_Q(t) + n_I(t)\sin(2\pi\alpha t) + n_Q(t)\cos(2\pi\alpha t)\big]\sin(4\pi f_c t) \end{aligned}$$

The output of the low-pass filter is given by

$$u(t) = \frac{1}{2}\big[x_I(t) + n_I(t)\cos(2\pi\alpha t) - n_Q(t)\sin(2\pi\alpha t)\big] \quad \text{(explain)}$$

Note that for AM, the actual demodulator includes a DC block after the low-pass filter, so the demodulator output for AM is

$$u^{'}(t) = \frac{A_c}{2}k_a m(t) + \frac{1}{2}n_I(t) = \frac{A_c}{2}\mu m_n(t) + \frac{1}{2}n_I(t)$$

since $f_0 = f_c \quad \Longrightarrow \alpha = 0$.

The output signal power is obtained by assuming $n_W(t) = 0$ thus $n(t) = n_I(t) = n_Q(t) = 0$.

$$\begin{aligned} u(t) &= \frac{1}{2}x_I(t)\alpha t) \\ u^{'}(t) &= \frac{A_c}{2}k_a m(t) = \frac{A_c}{2}\mu m_n(t) \quad \text{AM} \end{aligned}$$

The output signal power is given by

$$\begin{aligned} P_{S_O} &= \frac{R_{X_I}(0)}{4} \quad \text{DSB-SC,SSB, VSB} \\ P_{S_O} &= \frac{A_c^2 k_a^2}{4}P_M = \frac{A_c^2\mu^2}{4}P_{M_n} \quad \text{AM} \end{aligned}$$

e) Output noise power

Let us consider separately AM, DSB-SC and SSB, VSB.

For AM and DSB-SC, $f_0 = f_c$ so $\alpha = 0$. To find the output noise power we assume $m(t) = 0$, therefore

$$\begin{aligned} u(t) &= \frac{1}{2}n_I(t) \quad \text{DSB-SC} \\ u^{'}(t) &= \frac{1}{2}n_I(t) \quad \text{AM} \end{aligned}$$

The output noise power is given by

$$P_{N_0} = \frac{R_{N_I}(0)}{4} = \frac{N_0 W}{2}$$

For SSB and VSB, $\alpha \neq 0$,

$$u(t) = \frac{1}{2}\big[n_I(t)\cos(2\pi\alpha t) - n_Q(t)\sin(2\pi\alpha t)\big] = \frac{1}{2}\big[n_I^{'}(t) - n_Q^{'}(t)\big]$$

Since $N_I(t)$ and $N_Q(t)$ are zero mean WSS stationary, let us show that $N_I^{'}(t)$ and $N_Q^{'}(t)$ are zero mean wide-sense cyclo stationary and uncorrelated.

$$\begin{aligned} E\left[N_I^{'}(t)\right] &= E\left[N_I(t)\right]\cos(2\pi\alpha t) = 0 \\ E\left[N_Q^{'}(t)\right] &= E\left[N_Q(t)\right]\sin(2\pi\alpha t) = 0 \end{aligned}$$

$$
\begin{aligned}
R_{N_I'}(t+\tau,t) &= E\left[N_I'(t+\tau)N_I'(t)\right] \\
&= R_{N_I}(\tau)\cos(2\pi\alpha(t+\tau))\cos\big(2\pi\alpha t\big) \quad \text{periodic with period } \frac{1}{2\alpha} \\
R_{N_Q'}(t+\tau,t) &= E\left[N_Q'(t+\tau)N_Q'(t)\right] \\
&= R_{N_Q}(\tau)\sin\big(2\pi\alpha(t+\tau)\big)\sin\big(2\pi\alpha t\big) \quad \text{periodic with period } \frac{1}{2\alpha} \\
R_{N_I'N_Q'}(t+\tau,t) &= E\left[N_I'(t+\tau)N_Q'(t)\right] \\
&= R_{N_IN_Q}(\tau)\cos\big(2\pi\alpha(t+\tau)\big)\sin\big(2\pi\alpha t\big) \\
&= 0 \quad \text{since } N_I(t) \text{ and } N_Q(t) \text{ are uncorrelated and zero mean}
\end{aligned}
$$

Their "average" autocorrelation functions are given by

$$
\begin{aligned}
R^a_{N_I'}(\tau) &= 2\alpha\int_{-\frac{1}{4\alpha}}^{-\frac{1}{4\alpha}} R_{N_I'}(t+\tau,t)dt = \frac{R_{N_I}(\tau)}{2}\cos(2\pi\alpha\tau) \\
R^a_{N_Q'}(\tau) &= \frac{R_{N_Q}(\tau)}{2}\cos(2\pi\alpha\tau) = \frac{R_{N_I}(\tau)}{2}\cos(2\pi\alpha\tau) = R^a_{N_I'}(\tau)
\end{aligned}
$$

Since $N_I'(t)$ and $N_Q'(t)$ are uncorrelated and wide-sense stationary, $u(t)$ in absence of message signal is also wide-sense cyclo stationary. Its autocorrelation function at (t,t) is given by

$$
\begin{aligned}
R_U(t,t) = E\left[U^2(t)\right] &= \frac{1}{4}E\left[N_I'^2(t)\right] + \frac{1}{4}E\left[N_I'^2(t)\right] - \frac{1}{4}E\left[N_I'(t)N_Q'(t)\right] \\
&= \frac{1}{4}E\left[N_I'^2(t)\right] + \frac{1}{4}E\left[N_I'^2(t)\right]
\end{aligned}
$$

Hence the output noise power is

$$
P_{N_O} = R^a_U(0) = \frac{1}{4}R^a_{N_I'}(0) + \frac{1}{4}R^a_{N_I'}(0) = \frac{1}{2}R^a_{N_I'}(0) = \frac{R_{N_I}(0)}{4} = \frac{N_0B_T}{4}
$$

Exercise: Find the power spectral density of N_I' and N_Q' for SSB and verify that the output noise power for SSB is given by

$$
P_{N_O} = \frac{N_0W}{4}
$$

f) SNR's

AM:

$$\begin{aligned}
\mathrm{SNR}_I &= \frac{\mathrm{SNR}_C}{2} = \frac{A_c^2\left(1+k_a^2 P_M\right)}{4N_0W} = \frac{A_c^2\left(1+\mu^2 P_{M_n}\right)}{4N_0W} \\
\mathrm{SNR}_O &= \frac{A_c^2 k_a^2 P_M}{2N_0W} = \frac{A_c^2 \mu^2 P_{M_n}}{2N_0W} \\
\mathrm{SNR}_O &= \frac{2\mu^2 P_{M_n}}{1+\mu^2 P_{M_n}} \mathrm{SNR}_I = 2\nu \mathrm{SNR}_I \\
&= \nu \mathrm{SNR}_C
\end{aligned}$$

where ν ($\nu \leq \frac{1}{2}$) is the modulation efficiency given by

$$\nu = \frac{k_a^2 P_M}{1+k_a^2 P_M} = \frac{\mu^2 P_{M_n}}{1+\mu^2 P_{M_n}}$$

DSB-SC:

$$\begin{aligned}
\mathrm{SNR}_I &= \frac{\mathrm{SNR}_C}{2} = \frac{A_c^2 P_M}{4N_0W} \\
\mathrm{SNR}_O &= \frac{\frac{A_c^2}{4} P_M}{\frac{N_0W}{2}} = \frac{A_c^2 P_M}{2N_0W} \\
\mathrm{SNR}_O &= 2\mathrm{SNR}_I \\
&= \mathrm{SNR}_C
\end{aligned}$$

SSB:

$$\begin{aligned}
\mathrm{SNR}_I &= \mathrm{SNR}_C = \frac{A_c^2 P_M}{4N_0W} \\
\mathrm{SNR}_O &= \frac{\frac{A_c^2 P_M}{16}}{\frac{N_0W}{4}} = \frac{A_c^2 P_M}{4N_0W} \\
\mathrm{SNR}_O &= \mathrm{SNR}_I \\
&= \mathrm{SNR}_C
\end{aligned}$$

5.4 SNR's for AM with envelope detection

Draw a block diagram of a noisy receiver model with envelope detection.

The output of the bandpass filter is given by

$$z(t) = x(t) + n(t) = A_c\left(1 + k_a m(t)\right)\cos(2\pi f_c t) + n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t)$$

The complex envelope of $z(t)$ is given by

$$\tilde{z}(t) = A_c\left(1 + k_a m(t)\right) + n_I(t) + j n_Q(t)$$

The output of the envelope detector is

$$v(t) = |\tilde{z}(t)| = \left[\left(A_c\left(1 + k_a m(t)\right) + n_I(t)\right)^2 + n_Q^2(t)\right]^{1/2}$$

a) high SNR $A_c\left(1 + k_a m(t)\right) \gg |n_I(t)|, |n_Q(t)|$

The output of the envelope detector is given by

$$\begin{aligned} v(t) &= \left(A_c\left(1 + k_a m(t)\right) + n_I(t)\right)\left[1 + \left(\frac{n_Q(t)}{A_c\left(1 + k_a m(t)\right) + n_I(t)}\right)^2\right]^{1/2} \\ &\approx A_c\left(1 + k_a m(t)\right) + n_I(t) \end{aligned}$$

The output of the DC block is given by

$$u^{'}(t) = A_c k_a m(t) + n_I(t)$$

Hence the SNR's are given by

$$\begin{aligned} \mathrm{SNR}_I &= \frac{P_R}{P_N} = \frac{\frac{A_c^2}{2}\left(1 + k_a^2 P_M\right)}{2N_0 W} = \frac{A_c^2\left(1 + \mu^2 P_{M_n}\right)}{4N_0 W} \\ \mathrm{SNR}_O &= \frac{P_{S_O}}{P_{N_O}} = \frac{A_c^2 k_a^2 P_M}{2N_0 W} = \frac{A_c^2 \mu^2 P_{M_n}}{2N_0 W} \\ &= 2\nu \mathrm{SNR}_I = \nu \mathrm{SNR}_C \end{aligned}$$

(same result as for coherent detection)

b) low SNR $A_c\left(1 + k_a m(t)\right) \ll |n_I(t)|, |n_Q(t)|$

Show that in that case the output of the envelope detector is badly corrupted by noise.

5.5 SNR's for angle modulation with noise

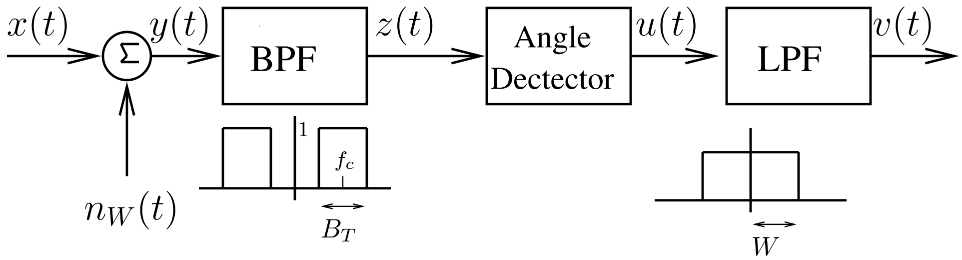

Figure 18: Noisy receiver model for angle modulation

$$
\begin{aligned}
z(t) &= x(t) + n(t) \\
&= A_c \cos\big(2\pi f_c t + \varphi(t)\big) + n_I(t)\cos(2\pi f_c t) - n_Q(t)\sin(2\pi f_c t) \\
\varphi(t) &= K \int_{-\infty}^{t} m(\tau) h(t-\tau) d\tau
\end{aligned}
$$

Assuming no noise, we have

$$v(t) = K^{'} m(t) \quad \text{(output of angle detector is proportional to } m(t)\text{)}$$

The lowpass filter is used to cut noise outside the frequency band of the recovered message signal.

a) Channel signal-to-noise ratio: SNR_C

Input signal power:

Assume first that $m(t)$ is deterministic, then the input signal power

$$
\begin{aligned}
P_x &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} A_c^2 \cos^2\big(2\pi f_c t + \varphi(t)\big) dt \\
&= \lim_{T\to\infty} \left[\frac{A_c^2}{2} \frac{1}{2T} \int_{-T}^{T} dt + \frac{A_c^2}{2} \frac{1}{2T} \int_{-T}^{T} \cos\big(4\pi f_c t + 2\varphi(t)\big) dt \right] \\
&= \frac{A_c^2}{2} + \frac{A_c^2}{2} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \cos\big(4\pi f_c t + 2\varphi(t)\big) dt
\end{aligned}
$$

Assume that $f_c \gg 1$ and that $\varphi(t)$ is slowly varying with respect to f_c, then $4\pi f_c t + 2\varphi(t) \approx 4\pi f_c t + 2\varphi_0$ and

$$P_x \approx \frac{A_c^2}{2} + \frac{A_c^2}{2} \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \cos\big(4\pi f_c t + 2\varphi_0\big) dt$$

$$
\begin{aligned}
&\approx \frac{A_c^2}{2} + \frac{A_c^2}{2} \lim_{T\to\infty} \frac{1}{2T}\frac{1}{4\pi f_c}\left[\sin\bigl(4\pi f_c T + 2\varphi_0\bigr) + \sin\bigl(4\pi f_c T - 2\varphi_0\bigr)\right] \\
&\approx \frac{A_c^2}{2}
\end{aligned}
$$

Assume now that $m(t)$ is the realization (sample function) of a stationary zero mean Gaussian random process $M(t)$ whose autocorrelation function is $R_M(\tau)$. The angle modulated signal $X(t)$ is now a random process which is neither wide-sense stationary nor wide-sense cyclo-stationary. However using generalization of Wiener-Khinchin theorem, its average autocorrelation function is defined as

$$
\begin{aligned}
R_X^a(\tau) &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R_X(t+\tau, t)dt \\
&= A_c^2 \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E\left[\cos\bigl(2\pi f_c(t+\tau) + \varphi(t+\tau)\bigr)\cos\bigl(2\pi f_c t + \varphi(t)\bigr)\right] dt \\
&= \frac{A_c^2}{2} \lim_{T\to\infty} \left\{ \frac{1}{2T}\int_{-T}^{T} E\left[\cos\bigl(2\pi f_c(2t+\tau) + \varphi(t+\tau) + \varphi(t)\bigr)\right] dt \right. \\
&\quad \left. + \frac{1}{2T}\int_{-T}^{T} E\left[\cos\bigl(2\pi f_c\tau + \varphi(t+\tau) - \varphi(t)\bigr)\right] dt \right\} \\
&= \frac{A_c^2}{2} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} E\left[\cos\bigl(2\pi f_c\tau + \varphi(t+\tau) - \varphi(t)\bigr)\right] dt \\
&= \frac{A_c^2}{2} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \Re\left\{ e^{j2\pi f_c\tau} E\left[e^{j[\varphi(t+\tau)-\varphi(t)]}\right]\right\} dt
\end{aligned}
$$

Since $\varphi(t)$ is the output of a linear time invariant filter with input $m(t)$, $\varphi(t)$ is also a stationary Gaussian random process with autocorrelation function $R_\varphi(\tau)$. It can be shown that for fixed t, $Z(t) = \varphi(t+\tau) - \varphi(t)$ is a zero-mean Gaussian random variable with variance

$$
\sigma_Z^2 = 2R_\varphi(0) - 2R_\varphi(\tau)
$$

Therefore

$$
E\left[e^{jZ(t)}\right] = E\left[e^{j\omega Z(t)}\right]\Big|_{\omega=1} = e^{-\frac{1}{2}\omega^2\sigma_Z^2}\Big|_{\omega=1} = e^{-[R_\varphi(0)-R_\varphi(\tau)]}
$$

And the average autocorrelation function of $X(t)$ is given by

$$
\begin{aligned}
R_X^a(\tau) &= \frac{A_c^2}{2} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \Re\left\{ e^{j2\pi f_c\tau} e^{-(R_\varphi(0)-R_\varphi(\tau))}\right\} dt \\
&= \frac{A_c^2}{2}\cos(2\pi f_c\tau) a(\tau)
\end{aligned}
$$

with

$$a(\tau) = E\left[e^{j(\varphi(t+\tau)-\varphi(t))}\right] = e^{-[R_\varphi(0)-R_\varphi(\tau)]}$$

The power of $X(t)$ is given by

$$P_X = R_X^a(0) = \frac{A_c^2}{2}$$

identical to the expression found when $m(t)$ is deterministic.

Input noise power:

$$P_N = N_0 B_T$$

Noise power in the message bandwidth:

$$P_{N'} = N_0 W$$

Input signal-to-noise ratio:

$$\mathrm{SNR}_I = \frac{A_c^2}{2N_0 B_T}$$

Channel signal-to-noise ratio:

$$\mathrm{SNR}_C = \frac{A_c^2}{2N_0 W}$$

b) Output signal-to-noise ratio assuming a high Carrier-to-noise ratio: SNR_O

High carrier-to-noise ratio is defined as:

$$A_c \gg r_n(t) = \sqrt{n_I^2(t) + n_Q^2(t)}$$

The angle detector and the lowpass filter can be modeled as illustrated in Fig. 19.

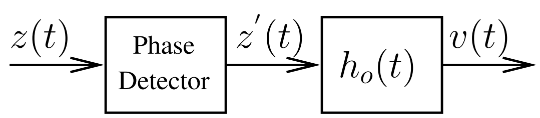

Figure 19: Noisy receiver model for angle modulation

where the filter $h_o(t)$ has transfer function given by

$$H_o(f) = \begin{cases} \frac{1}{H(f)}, & |f| < W \\ 0, & \text{else.} \end{cases}$$

The phase detector yields an output $\gamma\varphi_z(t)$ which is proportional to the phase of its input, where $\varphi_z(t)$ is the phase of $z(t)$ defined as

$$\varphi_z(t) = \arg\left[\tilde{z}(t)\right]$$

and $\tilde{z}(t)$ is the complex envelope of $z(t)$. For simplicity, let us assume that the proportionality constant is equal to 1. The output of the phase detector is given by

$$\begin{aligned}\varphi_z(t) &= \varphi(t) + \epsilon(t) \\ &= Km(t) * h(t) + \epsilon(t)\end{aligned}$$

The output of the filter $h_o(t)$ is given by

$$v(t) = Km(t) * h(t) * h_o(t) + \epsilon(t) * h_o(t)$$

Explain the chosen transfer function of $h_o(t)$ by considering first a noiseless case. Assume no noise, i.e. $n(t) = 0$, hence

$$\begin{aligned}z(t) &= \\ \varphi_z(t) &= \\ v(t) &= \end{aligned}$$

What is the desired output $v(t)$?

Consider now the noisy case again, then

$$v(t) = Km(t) * h(t) * h_o(t) + n_o(t) \tag{18}$$

where

$$n_o(t) = \epsilon(t) * h_o(t)$$

Hence from (18) we have the equivalent baseband model illustrated in Fig. 20.

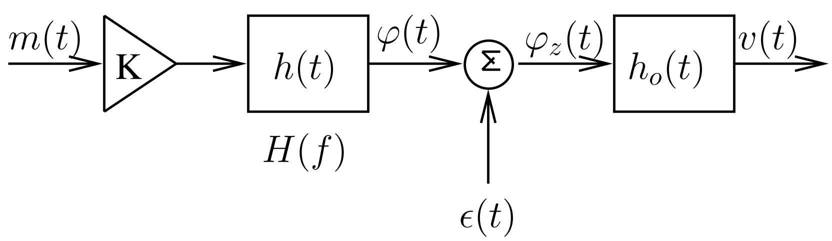

Figure 20: Noisy receiver equivalent model with angle modulation.

Expression of $\epsilon(t)$*:*

$$\begin{aligned}\tilde{z}(t) &= \tilde{x}(t) + \tilde{n}(t) = A_c e^{j\varphi(t)} + r_n(t) e^{j\theta_n(t)} \\ \tilde{z}(t) &= |\tilde{z}(t)| e^{j\varphi_z(t)} \\ \varphi_z(t) &= \varphi(t) + \epsilon(t)\end{aligned}$$

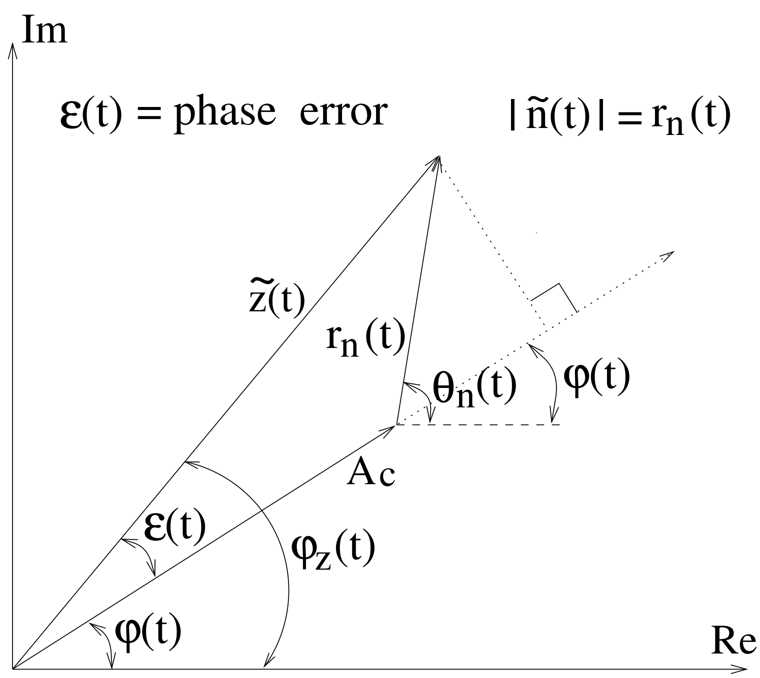

Figure 21: Phasor diagram for angle modulation with noise

Using the phasor diagram from Fig.21, show that

$$\tan\epsilon(t) = \frac{r_n(t)\sin\big(\theta_n(t) - \varphi(t)\big)}{A_c + r_n(t)\cos\big(\theta_n(t) - \varphi(t)\big)}$$

Assuming high carrier-to-noise ratio

$$\tan\epsilon(t) \approx \frac{r_n(t)}{A_c}\sin\big(\theta_n(t) - \varphi(t)\big) \ll 1$$

Therefore

$$\tan\epsilon(t) \approx \epsilon(t)$$

$$\begin{aligned}
&\approx \frac{r_n(t)}{A_c}\sin\bigl(\theta_n(t)-\varphi(t)\bigr)\\
&= \frac{r_n(t)}{A_c}\sin\bigl(\theta_n(t)\bigr)\cos\varphi(t) - \frac{r_n(t)}{A_c}\cos\bigl(\theta_n(t)\bigr)\sin\varphi(t)\\
&= \frac{n_Q(t)}{A_c}\cos\varphi(t) - \frac{n_I(t)}{A_c}\sin\varphi(t)
\end{aligned}$$

$$E\left[\epsilon(t)\right] = 0$$

since $n_I(t)$ and $n_Q(t)$ are zero mean. Assume that $m(t)$ is the realization of a WSS random process $M(t)$ independent of $N(t)$, hence $\varphi(t)$ is a WSS process independent of $N(t)$. The autocorrelation of $\epsilon(t)$ is given by

$$\begin{aligned}
R_\epsilon(t+\tau,t) &= E\left[\epsilon(t+\tau)\epsilon(t)\right]\\
&= \frac{1}{A_c^2}\Bigl\{E\left[N_Q(t+\tau)N_Q(t)\right]E\left[\cos\bigl(\varphi(t+\tau)\bigr)\cos\bigl(\varphi(t)\bigr)\right]\\
&\quad - E\left[N_Q(t+\tau)N_I(t)\right]E\left[\cos\bigl(\varphi(t+\tau)\bigr)\sin\bigl(\varphi(t)\bigr)\right]\\
&\quad - E\left[N_I(t+\tau)N_Q(t)\right]E\left[\sin\bigl(\varphi(t+\tau)\bigr)\cos\bigl(\varphi(t)\bigr)\right]\\
&\quad + E\left[N_I(t+\tau)N_I(t)\right]E\left[\sin\bigl(\varphi(t+\tau)\bigr)\sin\bigl(\varphi(t)\bigr)\right]\Bigr\}\\
&= \frac{1}{A_c^2}\left\{\frac{R_{N_I}(\tau)}{2}\Bigl(E\left[\cos\bigl(\varphi(t+\tau)+\varphi(t)\bigr)\right] + E\left[\cos\bigl(\varphi(t+\tau)-\varphi(t)\bigr)\right]\Bigr)\right.\\
&\quad \left. + \frac{R_{N_Q}(\tau)}{2}\Bigl(E\left[\cos\bigl(\varphi(t+\tau)-\varphi(t)\bigr)\right] - E\left[\cos\bigl(\varphi(t+\tau)+\varphi(t)\bigr)\right]\Bigr)\right\}\\
&= \frac{1}{A_c^2}R_{N_I}(\tau)E\left[\cos\bigl(\varphi(t+\tau)-\varphi(t)\bigr)\right]\\
&= \frac{1}{A_c^2}R_{N_I}(\tau)\Re\left\{E\left[e^{j(\varphi(t+\tau)-\varphi(t))}\right]\right\} \quad \text{since } R_{N_IN_Q}(\tau)=0 \text{ and } R_{N_I}(\tau)=R_{N_Q}(\tau)\\
&= \frac{R_{N_I}(\tau)}{A_c^2}a(\tau) = R_\epsilon(\tau)
\end{aligned}$$

where

$$a(\tau) = \Re\left\{E\left[e^{j(\varphi(t+\tau)-\varphi(t))}\right]\right\}$$

Note that if $M(t)$ is Gaussian, $a(\tau) = e^{-[R_\varphi(0)-R_\varphi(\tau)]}$.

From Fig. 20, the output signal $v(t)$ is given by

$$v(t) = Km(t) + n_o(t)$$

where $n_o(t) = \epsilon(t) * h_o(t)$. Therefore the output signal power is given by

$$P_{S_O} = K^2 P_M$$

where P_M is the power of the message signal $m(t)$. The output noise power is

$$P_{N_O} = E\left[n_o^2(t)\right] = \int_{-\infty}^{\infty} S_{N_o}(f)df = \int_{-\infty}^{\infty} |H_o(f)|^2 S_\epsilon(f)df = \int_{-W}^{W} |H_o(f)|^2 S_\epsilon(f)df$$

since $H_o(f) = 0$ for $|f| > W$. Therefore we only need to find the expression of $S_\epsilon(f)$ for $|f| \le W$. Let

$$A(f) = \mathcal{F}\left\{a(\tau)\right\}$$

Exercise: Assume that $A(f) \approx 0$ for $|f| > \frac{B_T}{2}$ and draw a typical graph of $A(f)$.

Let us assume that $W \ll \frac{B_T}{2}$ and $A(f) \approx 0$ for $|f| > \frac{B_T}{2}$, then

$$\begin{aligned}
S_\epsilon(f) = \mathcal{F}\left\{R_\epsilon(\tau)\right\} &= \frac{1}{A_c^2} S_{N_I}(f) * A(f) \quad \text{since } R_\epsilon(\tau) = \frac{R_{N_I}(\tau)}{A_c^2} a(\tau) \\
&= \frac{N_0}{A_c^2} \int_{-\frac{B_T}{2}}^{\frac{B_T}{2}} A(f-\lambda)d\lambda \\
&= \frac{N_0}{A_c^2} \int_{-\frac{B_T}{2}+f}^{\frac{B_T}{2}+f} A(u)du \qquad (u = f - \lambda) \\
&\approx \frac{N_0}{A_c^2} \int_{-\frac{B_T}{2}}^{\frac{B_T}{2}} A(u)du \quad \text{assuming } |f| \le W \text{ and } W \ll \frac{B_T}{2} \\
&\approx \frac{N_0}{A_c^2} \int_{-\infty}^{\infty} A(u)du \quad \text{assuming that } |f| \le W \text{ and } A(f) \approx 0 \text{ for } |f| > \frac{B_T}{2} \\
&= \frac{N_0}{A_c^2} a(0) = \frac{N_0}{A_c^2} \quad |f| \le W
\end{aligned}$$

Therefore since $H_o(f) = \frac{1}{H(f)}$ for $|f| \le W$, the output noise power is given by

$$P_{N_O} = \int_{-W}^{W} |H_o(f)|^2 S_\epsilon(f)df = \frac{N_0}{A_c^2} \int_{-W}^{W} \frac{df}{|H(f)|^2}$$

Output signal-to-noise ratio:

$$\text{SNR}_O = \frac{K^2 P_M}{\frac{N_0}{A_c^2} \int_{-W}^{W} \frac{df}{|H(f)|^2}}$$

$$
\begin{aligned}
&= \frac{2K^2 P_M W}{\int_{-W}^{W} \frac{df}{|H(f)|^2}} \mathrm{SNR}_C \qquad \text{using } \mathrm{SNR}_C = \frac{A_c^2}{2N_0 W} \\
&= \frac{2\beta^2 P_M W}{|H(W)|^2 \left(\max|m(t)|\right)^2 \int_{-W}^{W} \frac{df}{|H(f)|^2}} \mathrm{SNR}_C \qquad \text{using } \beta = K|H(W)| \max|m(t)| \\
&= \lambda \beta^2 P_{M_n} \mathrm{SNR}_C \\
&= 2\lambda(\beta+1)\beta^2 P_{M_n} \mathrm{SNR}_I \qquad \text{using } B_T = 2W(\beta+1) \text{ and } \mathrm{SNR}_C = \frac{B_T}{W}\mathrm{SNR}_I
\end{aligned}
$$

where λ is defined as

$$
\lambda = \left\{ \frac{|H(W)|^2}{2W} \int_{-W}^{W} \frac{df}{|H(f)|^2} \right\}^{-1}
$$

and $P_{M_n} = \frac{P_M}{(\max|m(t)|)^2}$ is the average-to-peak power ratio of the message signal or equivalently the power content of the normalized message signal.

Example:

$$
m(t) = A_m \cos(2\pi f_m t) \quad A_m \geq 0 \qquad \Longrightarrow \qquad P_M = \frac{A_m^2}{2}
$$

The normalized message signal is

$$
m_n(t) = \frac{m(t)}{\max|m(t)|} = \cos(2\pi f_m t) \qquad \Longrightarrow \qquad P_{M_n} = \frac{1}{2}
$$

Frequency modulation:

$$
H(f) = \frac{1}{j2\pi f} \quad \text{neglecting}^2 \qquad \frac{\delta(f)}{2}
$$

Thus

$$
\begin{aligned}
\lambda &= \left\{ \left(\frac{1}{2\pi W} \right)^2 \frac{1}{2W} \int_{-W}^{W} (2\pi f)^2 \, df \right\}^{-1} \\
&= \left\{ \frac{1}{2W^3} \int_{-W}^{W} f^2 df \right\}^{-1} = 3
\end{aligned}
$$

Phase modulation: PM

$$
H(f) = 1 \qquad \Longrightarrow \qquad \lambda = 1
$$

[2] See part f of section 4.4 for the justification of the form of $H(f)$.

c) Output signal-to-noise ratio SNR_O assuming a low Carrier-to-noise ratio:

Low Carrier-to-noise ratio: $A_c \ll r_n(t)$

Let us write

$$\varphi_z(t) = \theta_n(t) + \epsilon^{'}(t)$$

Interchanging the role of $\tilde{x}(t)$ and $\tilde{n}(t)$, it can be shown that

$$\varphi_z(t) \approx \theta_n(t) + \frac{A_c}{r_n(t)} \sin\bigl(\varphi(t) - \theta_n(t)\bigr)$$

Exercise: Draw a phasor digram of $\tilde{z}(t)$ at two time instants t_1 and t_2 such that the phase of the noise $\theta_n(t_2) \approx \theta_n(t_1) + 2\pi$. Then the phase $\varphi_z(t)$ will also change by 2π since it is dominated by the phase $\theta_n(t)$.

If during an interval $[t_1, t_2]$, $\varphi_z(t)$ changes by 2π, the output signal may have a rapid degradation during $[t_1, t_2]$ causing what is called FM clicks for FM modulation. For FM, $H_o(f) = j2\pi f$ corresponding to a differentiator.

$$v(t) = Km(t) + \frac{d\epsilon(t)}{dt}$$

Exercise: draw $\frac{d\epsilon(t)}{dt}$ for an FM modulated signal when due to the noise $\varphi_z(t)$ changes by 2π during certain intervals corresponding to low-carrier-to-noise situations.

We define a threshold value of SNR_C such that the high-carrier-to-noise ratio SNR_O formula applies. The threshold is defined as the minimum SNR_C or equivalently carrier-to-noise ratio SNR_I yielding an output signal-to-noise ratio that is not significantly deteriorated from the value predicted by the usual output signal-to-noise ratio formula.

Threshold SNR_C value for FM:

$$\mathrm{SNR}_{CT} = 20(\beta + 1)$$

$$\text{SNR}_{IT} = 10$$

In [1, p. 338], $\text{SNR}_{IT} = 20$ to correspond to practical observations, but the other threshold value is more often used in text books.

FM with tone modulation:

For FM with tone modulation, i.e.

$$m(t) = \cos(2\pi f_m t)$$

it can be shown that

$$\text{SNR}_O = \frac{\frac{3}{2}\beta^2 \text{SNR}_C}{1 + \frac{12\beta}{\pi}\text{SNR}_C \exp\left\{-\frac{1}{2(\beta+1)}\text{SNR}_C\right\}} \tag{19}$$

If high carrier-to-noise ratio is assumed in (19), then we obtain

$$\text{SNR}_O \approx \frac{3}{2}\beta^2 \text{SNR}_C$$

which agrees with the high carrier-to-noise ratio formula found previously ($\lambda = 3$ and $P_{M_n} = \frac{1}{2}$).